Homework 9 solutions

15.6 #15 Here $T = \{(x,y,z) | 0 \le x \le 1, 0 \le y \le 1-x, 0 \le z \le 1-x-y\}$, so

$$\int\int\int_T x^2\,dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} x^2\,dz\,dy\,dx = \int_0^1\int_0^{1-x} x^2(1-x-y)\,dy\,dx$$

$$= \int_0^1\int_0^{1-x^2}(x^2-x^3-x^2y)\,dy\,dx = \int_0^1 [x^2y - x^3y - \frac{1}{2}x^2y^2]_{y=0}^{y=1-x}\,dx$$

$$= \int_0^1 [x^2(1-x) - x^3(1-x) - \frac{1}{2}x^2(1-x)^2]\,dx$$

$$= \int_0^1 (\frac{1}{2}x^4 - x^3 + \frac{1}{2}x^2)\,dx = [\frac{1}{10}x^5 - \frac{1}{4}x^4 + \frac{1}{6}x^3]_0^1$$

$$= \frac{1}{10} - \frac{1}{4} + \frac{1}{6} = \frac{1}{60}$$

15.7 #22 In cylindrical coordinates E is the solid region within the cylinder $r = 1$ bounded above and below by the sphere $r^2 + z^2 = 4$, so $E = \{(r,\theta,z) | 0 \le \theta \le 2\pi, 0 \le r \le 1, -\sqrt{4-r^2} \le z \le \sqrt{4-r^2}\}$. Thus the volume is

$$\int\int\int_E dV = \int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 2r\sqrt{4-r^2}\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 2r\sqrt{4-r^2}\,dr = 2\pi[-\frac{2}{3}(4-r^2)^{\frac{3}{2}}]_0^1 = \frac{4}{3}\pi(8-3^{\frac{3}{2}}).$$

15.7 #28 The region of integration is the region above the plane $z = 0$ and below the paraboloid $z = 9 - x^2 - y^2$. Also, we have $-3 \le x \le 3$ with $0 \le y \le \sqrt{9-x^2}$ with describes the upper half of a circle of radius 3 in the

xy-plane centered at $(0,0)$. Thus,

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx = \int_{0}^{\pi}\int_{0}^{3}\int_{0}^{9-r^2}\sqrt{r^2}r\,dz\,dr\,d\theta$$

$$= \int_{0}^{\pi}\int_{0}^{3}\int_{0}^{9-r^2} r^2\,dz\,dr\,d\theta = \int_{0}^{\pi}\int_{0}^{3} r^2(9-r^2)\,dr\,d\theta$$

$$= \int_{0}^{\pi} d\theta \int_{0}^{3}(9r^2-r^4)\,dr = [\theta]_0^\pi[3r^3-\frac{1}{5}r^5]_0^3$$

$$= \pi(81-\frac{243}{5}) = \frac{162}{5}\pi = 32.4\pi$$

15.8 #14 $\rho \leq 2$ represents the solid sphere of radius 2 centered at the origin. Notice that $x^2+y^2 = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 = \rho^2\sin^2\phi$. Then $\rho = \csc\phi \Rightarrow r = \rho\sin\phi = 1 \Rightarrow \rho^2\sin^2\phi = x^2+y^2 = 1$ so $\rho \leq \csc\phi$ restricts the solid to that portion on or inside the circular cylinder $x^2+y^2=1$.

15.8 #15 We use the inequality $z \geq \sqrt{x^2+y^2}$ because the solid lies above the cone. Squaring both sides of this inequality gives $z^2 \geq x^2+y^2 \Rightarrow 2z^2 \geq x^2+y^2+z^2 = \rho^2 \Rightarrow z^2 = \rho^2\cos^2\phi \geq \frac{1}{2}\rho^2 \Rightarrow \cos^2\phi \geq \frac{1}{2}$. The cone opens upward so that the inequality is $\cos\phi \geq \frac{1}{\sqrt{2}}$, or equivalently $0 \leq \phi \leq \frac{\pi}{4}$. One could also use the fact that the cone is $z = r$ and so in spherical coordinates $\rho\cos\phi = \rho\sin\phi \Rightarrow \cos\phi = \sin\phi \Rightarrow \phi = \pi/4$.

In spherical coordinates the sphere $z = x^2+y^2+z^2$ is $\rho\cos\phi = \rho^2 \rightarrow \rho = \cos\phi$. $0 \leq \rho \leq \cos\phi$ because the solid lies below the sphere. The solid can there fore be described as the region in spherical coordinates satisfying $0 \leq \rho \leq \cos\phi, 0 \leq \phi \leq \frac{\pi}{4}$.