Homework 6 solutions

§14.7 #11 $f(x,y) = x^3 - 12xy + 8y^3 \Rightarrow f_x = 3x^2 - 12y, f_y = -12x + 24y^2$, $f_{xx} = 6x, f_{xy} = -12, f_{yy} = 48y$.

Then, $f_x = 0$ implies $x^2 = 4y$ and $f_y = 0$ implies $x = 2y^2$. Substituting the second equation into the first gives $(2y^2)^2 = 4y \Rightarrow 4y^4 = 4y \Rightarrow 4y(y^3 - 1) = 0 \Rightarrow y = 0$ or $y = 1$.

If $y = 0$ then $x = 0$ and if $y = 1$ then $x = 2$, so the critical points are $(0,0)$ and $(2,1)$.

$D(0,0) = (0)(0) - (-12)^2 = -144 < 0$, so $(0,0)$ is a saddle point. $D(2,1) = (12)(48) - (-12)^2 = 432 > 0$ and $f_{xx}(2,1) = 12 > 0$ so $f(2,1) = -8$ is a local minimum.

§14.7 #35 $f_x(x,y) = 6x^2$ and $f_y(x,y) = 4y^3$. And so $f_x = 0$ and $f_y = 0$ only occur when $x = y = 0$. Hence, the only critical point inside the disk is at $x = y = 0$ where $f(0,0) = 0$. Now on the circle $x^2 + y^2 = 1$, $y^2 = 1 - x^2$ so let $g(x) = f(x,y) = 2x^3 + (1 - x^2)^2 = x^4 + 2x^3 - 2x^2 + 1$, $-1 \leqslant x \leqslant 1$. Then $g'(x) = 4x^3 + 6x^2 - 4x = 0 \Rightarrow x = 0, -2$, or $\frac{1}{2}$. $f(0, \pm 1) = g(0) = 1, f(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}) = g(\frac{1}{2}) = \frac{13}{16}$, and $(-2,-3)$ is not in D. Checking the endpoints, we get $f(-1,0) = g(-1) = -2$ and $f(1,0) = g(1) = 2$. Thus the absolute maximum and minimum of f on D are $f(1,0) = 2$ and $f(-1,0) = -2$.

Another method: On the boundary $x^2 + y^2 = 1$ we can write $x = \cos\theta$, $y = \sin\theta$, so $f(\cos\theta, \sin\theta) = 2\cos^3\theta + \sin^4\theta$, $0 \le \theta \le 2\pi$.

§14.8 #6 $f(x,y) = e^{xy}$, $g(x,y) = x^3 + y^3 = 16$, and $\nabla f = \lambda \nabla g \Rightarrow \langle ye^{xy}, xe^{xy}\rangle = \langle 3\lambda x^2, 3\lambda y^2\rangle$, so $ye^{xy} = 3\lambda x^2$ and $xe^{xy} = 3\lambda y^2$.

Note that $x = 0 \Leftrightarrow y = 0$ which contradicts $x^3 + y^3 = 16$, so we may assume $x \neq 0$, $y \neq 0$, and then $\lambda = \frac{ye^{xy}}{3x^2} = \frac{xe^{xy}}{3y^2} \Rightarrow x^3 = y^3 \Rightarrow x = y$. But $x^3 + y^3 = 16$, so $2x^3 = 16 \Rightarrow x = 2 = y$. Here there is no minimum value, since we can choose points (e.g. with x huge and positive and y huge and negative) satisfying the constraint $x^3 + y^3 = 16$ that make $f(x, y) = e^{xy}$ arbitrarily close to 0 (but never equal to 0). Note that the constraint curve is not bounded. The maximum value is $f(2, 2) = e^4$.

§14.8 # 10 Use Lagrange multipliers to find the maximum and minimum values of $f(x, y, z) = x^2y^2z^2$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

We need $\nabla f = \lambda \nabla g \Rightarrow \langle 2xy^2z^2, 2x^2yz^2, 2x^2y^2z\rangle = \langle 2\lambda x, 2\lambda y, 2\lambda z\rangle$. Then we have (1) $\lambda = y^2z^2 = x^2z^2 = x^2y^2$ and $\lambda \neq 0$, or (2) $\lambda = 0$ and one or two (but not three) of the coordinates are 0. If (1) holds, then $x^2 = y^2 = z^2 = \frac{1}{3}$. The minimum value of f on the sphere occurs in case (2) with a value of 0 and the maximum value is $\frac{1}{27}$ which arises from all the points from (1), that is, the points $(\pm\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, $(\pm\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, $(\pm\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$, and $(\pm\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$.

§14.8 #27 Let $f(x, y, z) = d^2 = (x - 2)^2 + (y - 1)^2 + (z + 1)^2$, then we want to minimize f subject to the constraint $g(x, y, z) = x + y - z = 1$. We need

$$\nabla f = \lambda \nabla g \Rightarrow \langle 2(x - 2), 2(y - 1), 2(z + 1)\rangle = \langle \lambda, \lambda, -\lambda\rangle.$$

So, $x = 0.5\lambda + 2$, $y = 0.5\lambda + 1$, and $z = -0.5\lambda - 1$. Substituting into the constraint equation gives $\frac{\lambda+4}{2} + \frac{\lambda+2}{2} + \frac{\lambda+2}{2} = 1 \Rightarrow 3\lambda + 8 = 2 \Rightarrow \lambda = -2$, so $x = 1$, $y = 0$, and $z = 0$. This must correspond to a minimum, so the shortest distance is

$$d = \sqrt{(1 - 2)^2 + (0 - 1)^2 + (0 + 1)^2} = \sqrt{3}.$$