Homework 5 solutions

§14.4 #20 Find the linear approximation of the function $f(x, y) = \ln(x - 3y)$ at $(7, 2)$ and use it to approximate $f(6.9, 2.06)$.

Start by taking the partial derivatives of f with respect to x and y and then evaluate them at $(7, 2)$.

$$f_x(x, y) = \frac{1}{x - 3y} \Rightarrow f_x(7, 2) = 1,$$
$$f_y(x, y) = \frac{-3}{x - 3y} \Rightarrow f_y(7, 2) = -3.$$

Then the linear approximation of f at $(7, 2)$ is given by

$$L(x, y) = f(7, 2) + f_x(7, 2)(x - 7) + f_y(7, 2)(y - 2) = x - 3y - 1.$$

Thus $f(x, y) \approx L(x, y)$ when $(x, y) \approx (7, 2)$. Hence, $f(6.9, 2.06) \approx L(6.9, 2.06) = 6.9 - 3(2.06) - 1 = -0.28$.

§14.4 #34 The dimensions of a closed rectangular box are measured as 80cm, 60 cm, and 50 cm, respectively, with a possible error of 0.2 cm in each dimension. Use differentials to estimate the maximum error in the calculated surface area of the rectangular box.

Let S be the surface area, and let x, y, z be the respective height, length, width of the box. Then $S = 2(xy + xz + yz)$ and

$$dS = 2(y + z)dx + 2(x + z)dy + 2(x + y)dz.$$

The maximum error occurs with $\Delta x = \Delta y = \Delta z = 0.2$. Using $dx = \Delta x$, $dy = \Delta y$, and $dz = \Delta z$ (and $x = 80$, $y = 60$, and $z = 50$), we find the maximum error in the calculated surface area to be about

$$dS = (220)(0.2) + (260)(0.2) + (280)(0.2) = 152cm^2.$$

§14.5 #22 Use the Chain Rule to find the indicated partial derivatives.

Let $u = \sqrt{r^2 + s^2}$, $r = y + x\cos t$, and $s = x + y \sin t$. Then,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = (r\cos t + s)/\sqrt{r^2+s^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} = (r + s\sin t)/\sqrt{r^2+s^2}$$

and

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial t} = (sy\cos t - rx\sin t)/\sqrt{r^2+s^2}$$

When $x = 1$, $y = 2$, $t = 0$, we have $r = 3$ and $s = 1$, so that

$$\frac{\partial u}{\partial x} = \frac{4}{\sqrt{10}},$$

$$\frac{\partial u}{\partial y} = \frac{3}{\sqrt{10}},$$

and

$$\frac{\partial u}{\partial t} = \frac{2}{\sqrt{10}}.$$

§14.5 # 39 The length l, width w and height h of a box change with time. At a certain instant the dimensions are $l = 1m$, $w = h = 2m$, and l and w are increasing at a rate of $2m/s$ while h is decreasing at a rate of $3m/s$. At that instant find the rates at which the following quantities are changing.

(a) The volume.

$V = lwh$, so by the Chain Rule

$$\frac{dV}{dt} = \frac{\partial V}{\partial l}\frac{dl}{dt} + \frac{\partial V}{\partial w}\frac{dw}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = wh\frac{dl}{dt} + lh\frac{dw}{dt} + lw\frac{dh}{dt}$$

$$= 2\cdot 2\cdot 2 + 1\cdot 2\cdot 2 + 1\cdot 2\cdot(-3) = 6m^3/s.$$

(b) The surface area.

$S = 2(lw + lh + wh)$, so by the Chain Rule

$$\frac{dS}{dt} = \frac{\partial S}{\partial l}\frac{dl}{dt} + \frac{\partial S}{\partial w}\frac{dw}{dt} + \frac{\partial S}{\partial h}\frac{dh}{dt} = 2(w+h)\frac{dl}{dt} + 2(l+h)\frac{dw}{dt} + 2(l+w)\frac{dh}{dt}$$

$$2(2+2)2 + 2(1+2)2 + 2(1+2)(-3) = 10m^2/s.$$

(c) The length of a diagonal.

$L^2 = l^2 + w^2 + h^2$ so that by implicit differentiation we have

$$2L\frac{dL}{dt} = 2l\frac{dl}{dt} + 2w\frac{dw}{dt} + 2h\frac{dh}{dt}$$

$$= 2(1)(2) + 2(2)(2) + 2(2)(-3) = 0 \Rightarrow \frac{dL}{dt} = 0m/s.$$

§14.6 #22 Find the maximum rate of change of $f(p,q) = \; qe^{-p} + pe^{-q}$ at $(0,0)$ and the direction in which it occurs.

The maximum rate of change will occur in the same direction as the direction of

$$\nabla f(p,q) = \langle\; -qe^{-p} + e^{-q},\; e^{-p} - pe^{-q}\rangle \Rightarrow \nabla f(0,0) = \langle 1, 1\rangle.$$

Hence this direction (as a unit vector is)

$$\vec{u} = \langle 1/\sqrt{2}, 1/\sqrt{2}\rangle.$$

The maximum rate of change is $|\nabla f(0,0)| = \sqrt{2}$