# Homework 10 solutions

**15.9 #15** The Jacobian $J = \frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ 0 & 1 \end{pmatrix} = \frac{1}{v}$. Note that $y = x$ is the image of the parabola $v^2 = u$ and $y = 3x$ is the image of the parabola $v^2 = 3u$. Since $xy = u$, the hyperbolas $xy = 1, xy = 3$ are the images of the lines $u = 1$ and $u = 3$ respectively. Thus after considering the picture of the region in the $uv$-plane

$$\int\int_R xy\, dA = \int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} u(\frac{1}{v})\, dv\, du = \int_1^3 u(\ln\sqrt{3u} - \ln\sqrt{u})\, du$$

$$= \int_1^3 u \ln\sqrt{3}\, du = 4\ln\sqrt{3} = 2\ln 3.$$

**16.1 #21**

$$f(x,y) = xe^{xy} \Rightarrow \nabla f(x,y) = f_x(x,y)\vec{i} + f_y(x,y)\vec{j} = (xe^{xy}y + e^{xy})\vec{i} + (xe^{xy}x)\vec{j}$$

$$= (xy+1)e^{xy}\vec{i} + x^2 e^{xy}\vec{j}$$

**16.2 #7**

$$C = C_1 + C_2$$

$$\text{On } C_1 : x = t, y = 0, 0 \le t \le 2 \Rightarrow dx = dt,\ dy = 0\, dt.$$

$$\text{On } C_2 : x = t, y = 2t - 4, 2 \le t \le 3 \Rightarrow dx = dt,\ dy = 2\, dt.$$

Then

$$\int_c xy\, dx + (x-y)\, dy = \int_{c_1} xy\, dx + (x-y)\, dy + \int_{c_2} xy\, dx + (x-y)\, dy$$

$$= \int_0^2 (t)(0)\, dt + (t-0)(0)\, dt + \int_2^3 [(2t^2 - 4t) + (-t+4)(2)]\, dt$$

$$= \int_2^3 (2t^2 - 6t + 8)\, dt = \frac{17}{3}$$

**16.2 #18** Vectors starting on $C_1$ point in roughly the same direction as $C_1$, so $\vec{F}\cdot\vec{T}$ is positive there. Then $\int_{c_1} \vec{F} \cdot d\vec{r} = \int_{c_1} \vec{F} \cdot \vec{T}\, ds$ is positive. On the other hand, on $C_2$ only very small vectors point in the same direction as $C_2$, while some large vectors point in roughly the opposite direction, so we would expect $\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot \vec{T}\, ds$ to be negative.

**16.2 #20**

$$\vec{F}(\vec{r}(t)) = (t^2 + t^3)\vec{i} + (t^3 - t^2)\vec{j} + (t^2)^2\vec{k} = (t^2 + t^3)\vec{i} + (t^3 - t^2)\vec{j} + t^4\vec{k},$$

$$\vec{r}\,'(t) = 2t\,\vec{i} + 3t^2\,\vec{j} + 2t\,\vec{k}.$$

Then

$$\int_c F \cdot dr = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_0^1 (2t^3 + 2t^4 + 3t^5 - 3t^4 + 2t^5)\, dt = \int_0^1 (5t^5 - t^4 + 2t^3)\, dt$$

$$= [\frac{5}{6}t^6 - \frac{1}{5}t^5 + \frac{1}{2}t^4]_0^1 = \frac{5}{6} - \frac{1}{5} + \frac{1}{2} = \frac{17}{15}.$$